Higher secant varieties of minimal degree and del Pezzo secant varieties

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TMS Annual Meeting
January 18, 2022

Notions and projective invariants

- Projective complex variety X is a common zero locus of homogeneous equations $f_1, \ldots, f_m \subset R = \mathbb{C}[x_0, x_1, \cdots, x_N]$, i.e. $X := Z(f_1, \ldots, f_m) \subset \mathbb{P}^N = \mathbb{P}(V^*),\ \dim_{\mathbb{C}} V^* = N + 1, \dim(X) = n$.
- We assume that X is irreducible, not necessarily smooth.
- $I_X = \{f \in R \mid f(p) = 0, \forall p \in X\}$ is the defining ideal of X, and R/I_X is called "the projective coordinate ring" of X.
- $\deg(X)$ is defined as the number of finite points $X \cap L^{N-n}$ for general linear subspace L of dimension $N - n$.
- It is elementary to check that $\deg(X) \geq \operatorname{codim}(X) + 1 = N - n + 1$ and X is called a variety of minimal degree if

$$\deg(X) = \operatorname{codim}(X) + 1.$$

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Varieties of minimal degree

- Note that X is of minimal degree if and only if its general curve section is a rational normal curve.
- [The del Pezzo-Bertini classification, 1886 and 1907]

  $X \subset \mathbb{P}^N$ is of minimal degree if and only if X is (a cone of) one of the following:
  - a quadric hypersurface;
  - a Veronese surface $\nu_2(\mathbb{P}^2)$ in $\mathbb{P}^5$;
  - a rational normal scroll, i.e. $\mathbb{P}(\mathcal{E}) \hookrightarrow \mathbb{P}^{\Sigma a_i + d}$ where $\mathcal{E} \simeq \bigoplus_{i=0}^{d} \mathcal{O}_{\mathbb{P}^1}(a_i), a_i \geq 1$.

  ▶ See also the expository paper "On varieties of minimal degree (A centennial account)-1987" due to D. Eisenbud and J. Harris.

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del Pezzo varieties

- The sectional genus $\pi(X)$ of X is the arithmetic genus of a general hyperplain section curve of X when $\dim(X) \geq 2$.
- If $\deg X = \mathrm{codim} X + 2$, i.e. almost minimal degree case, then

$$\pi(X) = 0 \text{ or } 1.$$

X is called a del Pezzo variety if $\deg(X) = \mathrm{codim}(X) + 2$ and the sectional genus $\pi(X) = 1$, or equivalently $\mathrm{depth}(X) = n + 1$, i.e., X is arithmetically Cohen-Macaulay with almost minimal degree.

- Note that X is a del Pezzo variety if and only if its general curve section is either an elliptic normal curve, a rational curve with one cusp or rational nodal curve.

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Classification

- del Pezzo curves are
  - an elliptic normal curve;
  - a rational nodal curve or rational curve with one cusp
- Smooth del Pezzo surfaces are
  - $(\mathbb{P}^2, |3H - p_1 - \cdots - p_s|)$ in $\mathbb{P}^{9-s}$ with $0 \leq s \leq 6$; and
  - $(\mathbb{P}^1 \times \mathbb{P}^1, |\mathcal{O}(2,2)|)$ in $\mathbb{P}^8$.
- T. Fujita classified smooth del Pezzo varieties into 8 types:
  - a cubic hypersurface, complete intersection of type $(2,2)$,
  - $\mathbb{G}(1,4)$, $v_2(\mathbb{P}^3)$, a smooth del Pezzo surface,
  - $\mathbb{P}^2 \times \mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)) \hookrightarrow \mathbb{P}^8$.

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Generalizaton

In this talk, I'd like to introduce very similar objects in the category of higher secant varieties:

1. Define higher secant varieties of minimal degree.
2. Define del Pezzo higher secant varieties.
3. Many structural theorems like "Matryoshka" for higher secant varieties of minimal degree in Algebra and Geometry.
4. Finding more Matryoshka structures in the category of higher secant varieties.

♣ F. Zak and F. Russo used this word "Matryoshka" for the first time.

- "Matryoshka" is a set of traditional Russian dolls with the same shape but different sizes inside successively.
- Mathematically, many fundamental results for irreducible varieties are also expected to hold in the category of higher secant varieties in a very similar pattern.

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Higher secant varieties

Let $X \subset \mathbb{P}^N$ be an irreducible nondegenerate projective variety, and $q \geq 1$ a positive integer. We define the *q-secant variety*

$$\mathrm{Sec}^q(X) := \overline{\bigcup_{x_1,\ldots,x_q} \langle x_1, \ldots, x_q \rangle} \subseteq \mathbb{P}^N$$

where $x_1, \ldots, x_q$ are linearly independent points in X.

$X = \mathrm{Sec}^1(X) \subsetneq \cdots \subsetneq \mathrm{Sec}^q(X) \subsetneq \cdots \subsetneq \mathrm{Sec}^{k_0-1}(X) \subsetneq \mathrm{Sec}^{k_0}(X) = \mathbb{P}^N$.

- It is well known that $\mathrm{Sec}^{q-1}(X) \subseteq \mathrm{Sing}(\mathrm{Sec}^q(X))$.
- $\dim(X) + 2(q-1) \leq \dim(\mathrm{Sec}^q(X)) \leq \min\{qn + (q-1), N\}$. The right-hand side is called the *expected dimension* of $\mathrm{Sec}^q(X)$.
- X is q-defective (or $\mathrm{Sec}^q(X)$ is defective) if $\dim(\mathrm{Sec}^q(X)) < \min\{qn + (q-1), N\}$.

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Terracini's Lemma

Computing the exact dimension of higher secant varieties is important in both projective algebraic geometry and tensor geometry.

- **Theorem** (Terracini's Lemma(1911))
$X \subset \mathbb{P}^N$: a complex non-degenerate irreducible variety and $q \geq 1$. Then, $T_y\mathrm{Sec}^q(X) = \langle T_{x_1}X, \ldots, T_{x_q}X \rangle$ in $\mathbb{P}^N$ when $x_1, \ldots, x_q \in X$ are general points and $y \in \langle x_1, \ldots, x_q \rangle \subseteq \mathrm{Sec}^q(X)$ is also a general point. In particular, $\dim \mathrm{Sec}^q(X) = \dim \langle T_{x_1}X, \ldots, T_{x_q}X \rangle$ for general points $x_1, \ldots, x_q \in X$.

- Suppose $\mathrm{Sec}^q(X) \subsetneq \mathbb{P}^N$ and $\dim \mathrm{Sec}^q(X) = qn + (q-1)$. Then, all the tangent spaces $T_{x_1}X, \ldots, T_{x_q}X$ are all disjoint for general points $x_1, \ldots, x_q \in X$.

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- Suppose $\mathrm{Sec}^q(X) \subsetneq \mathbb{P}^N$ and $\dim \mathrm{Sec}^q(X) = qn + (q-1)$. Then, all the tangent spaces $T_{x_1} X, \ldots, T_{x_q} X$ are all disjoint for general points $x_1, \ldots, x_q \in X$.

The tangent cone of a higher secant variety

Note that $X \subseteq \mathrm{Sec}^{q-1}(X) \subseteq \mathrm{Sing}(\mathrm{Sec}^q(X))$.

It is helpful to understand the tangent cone $TC_z\mathrm{Sec}^q(X)$ at a general singular point $z \in X$ of $\mathrm{Sec}^q(X)$.

- **Theorem**(Ciliberto-Russo, 2006)
  Let z be a general singular point of $\mathrm{Sec}^q(X)$ contained in X. Then, the cone $J(T_z(X), \mathrm{Sec}^{q-1}(X_{T_z}))$ is an irreducible component of $TC_z(\mathrm{Sec}^q(X))_{\mathrm{red}}$ where X_{T_z} is the image of the tangential projection of X from $T_z(X)$ into $\mathbb{P}^{N-n-1}$. Therefore,
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New lower bound on degree

- $\dim \mathrm{Sec}^q(X) = \dim TC_z(\mathrm{Sec}^q(X)) = n + 1 + \dim \mathrm{Sec}^{q-1}(X_{T_z})$. So, $\mathrm{Sec}^q(X)$ and $\mathrm{Sec}^{q-1}(X_{T_z})$ have the same codimension.
- Degree formula
  $\deg(X) = \deg(\pi_z)\deg(X_z) + \deg TC_z(X)$ for any point $z \in X$ where $\pi_z : X \dashrightarrow X_q \subset \mathbb{P}^{n+e-1}$ is an inner projection.
- Theorem [Ciliberto-Russo, 2006]
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in the category of q-secant varieties.

- We call $\operatorname{Sec}^q(X)$ a q-secant variety of minimal degree if $\deg \operatorname{Sec}^q(X) = \binom{e+q}{q}$.

Natural questions are classification and characterization of higher secant varieties of minimal degree:

- Find a pair (X, q) such that $\operatorname{Sec}^q(X)$ is of minimal degree;
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Properties of secant varieties of minimal degree

- For a curce C in $\mathbb{P}^N$, it can be shown that $S^q(C)$ is of minimal degree if and only if C is of minimal degree.
- Let X be a variety of any dimenson in $\mathbb{P}^N$. If X is of minimal degree then $S^q(X)$ is of minimal degree.
  The converse is not true. For example, if S is a del Pezzo surface then $S^q(S)$ is of minimal degree.

Theorem (Ciliberto-Russo, Choe-K)

- $S^q(X)$ is of minimal degree if and only if $S^{q-1}(X_{\mathbb{T}_x(X)})$ is of minimal degree.
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Examples of higher secant varieties of minimal degree

- Rational normal scrolls with $q \geq 1$;
- smooth del Pezzo surfaces with $q \geq 2$;
- For 2-Veronese varieties $\nu_2(\mathbb{P}^n)$, $n \geq 2$, it can be checked that $\mathrm{Sec}^q(\nu_2(\mathbb{P}^n))$ with $q \geq n-1$ is of minimal degree.
- Segre varieties $\sigma(\mathbb{P}^a \times \mathbb{P}^b)$ with $q \geq \min\{a, b\}$.

For further examples, consult with [Ciliberto-Russo, 2006].

Remark

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Equations and linear relations

- Note that there is no equation of degree q vanishing on $\mathrm{Sec}^q(X)$.
- **The generalized $K_{p,1}$ theorem** (M. Green(84), Choe-K)

  Let $e = \mathrm{codim} S^q(X)$. Then,
  - $K_{p,q}(S^q(X)) = 0$ for every $p > e$; and
  - $K_{e,q}(S^q(X)) \neq 0$ if and only if $S^q(X)$ is of minimal degree.
- **The upper bound of $\dim K_{p,q}$** (Choe-K)
  - $\dim K_{p,q} := \beta_{p,q}(S^q(X)) \leq \binom{p+q-1}{q}\binom{e+q}{p+q}$;
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Let $\mathrm{Sec}^q(X) \subsetneq \mathbb{P}^N$, $e = \mathrm{codim}(Sec^q(X))$.

- There is no form of degree q vanishing on $\mathrm{Sec}^q(X)$ (elementary!).
- So, the q-th row is possibly the first nontrivial in the Betti table.

♣ **The Betti table of $Sec^q(X)$**

$j \backslash i$	0	1	2	3	$\cdots$	$i-1$	i	$i+1$	$\cdots$	$\triangle$
0	1	–	–	–	$\cdots$	–	–	–	$\cdots$	–
1	–	–	–	–	$\cdots$	–	–	–	$\cdots$	–
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$q-1$	–	–	–	–	$\cdots$	–	–	–	$\cdots$	–
q	–	$\beta_{1,q}$	$\beta_{2,q}$	$\beta_{3,q}$	$\cdots$	$\beta_{i-1,q}$	$\beta_{i,q}$	$\beta_{i+1,q}$	$\cdots$	$\beta_{\triangle,q}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
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del Pezzo objects in higher secant varieties

- It can be proved that if $\deg S^q(X) \neq \binom{e+q}{q}$, i.e. $\mathrm{Sec}^q(X)$ is not a q-secant variety of minimal degree, then $\deg(S^q(X)) \geq \binom{e+q}{q} + \binom{e+q-1}{q-1}$.
- Suppose that $\deg S^q(X) = \binom{e+q}{q} + \binom{e+q-1}{q-1}$. Then $\pi(S^q(X)) \leq (q-1)\deg S^q(X) + 1$.
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- **(Queston)** What kind of varieties have del Pezzo q-secant varieties for some $q \geq 1$? ($q = 1 \Rightarrow$ usual del Pezzo, $q \geq 2 \Rightarrow$ del Pezzo higher secant varieties)

del Pezzo objects in higher secant varieties

- It can be proved that if $\deg S^q(X) \neq \binom{e+q}{q}$, i.e. $\mathrm{Sec}^q(X)$ is not a q-secant variety of minimal degree, then $\deg(S^q(X)) \geq \binom{e+q}{q} + \binom{e+q-1}{q-1}$.
- Suppose that $\deg S^q(X) = \binom{e+q}{q} + \binom{e+q-1}{q-1}$. Then $\pi(S^q(X)) \leq (q-1)\deg S^q(X) + 1$.
- $S^q(X)$ is called a del Pezzo q-secant variety (Choe-K, 2021) if
  - $\deg S^q(X) = \binom{e+q}{q} + \binom{e+q-1}{q-1}$ and
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Examples of del Pezzo higher secant varieties

- Let $C \subset \mathbb{P}^N$ be an elliptic normal curve. Then, $\mathrm{Sec}^q(C)$ is a del Pezzo q-secant variety for all $q \geq 1$ if $\mathrm{Sec}^q(C) \subsetneq \mathbb{P}^N$. ([Bothmer-Hulek, 2004] or [Fisher, 2006]);
- Let $X = v_4(\mathbb{P}^2) \subset \mathbb{P}^{14}$ be the fourth Veronese surface. Then, $X \subsetneq S^2(X) \subsetneq S^3(X) \subsetneq S^4(X) \subsetneq S^5(X) = \mathbb{P}^{14}$.
  - $S^3(X)$ is del Pezzo of dimension 8
  - $S^4(X)$ is of minimal degree with dimension 11.
- Let $Y \subset \mathbb{P}^{11}$ be an embedding of $S(1,2) \subset \mathbb{P}^4$ by $|\mathcal{O}_{S(1,2)}(2)|$.

$$Y \subsetneq S^2(Y) \subsetneq S^3(Y) \subsetneq S^4(Y) = \mathbb{P}^{11}.$$

Then, $S^2(Y)$ is del Pezzo and $S^3(Y)$ is minimal.

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- Consider Plücker embedding $\mathbb{G}(1, 2q+2) \subset \mathbb{P}^N, N = \binom{2q+3}{2} - 1$ of the Grassmannian of lines. Then $S^q(X)$ is del Pezzo
- Then, a general tangential projection of X is $Y = \mathbb{G}(1, 2q) \subset \mathbb{P}^{N'}, N' = \binom{2q+1}{2} - 1$. Thus, $S^{q-1}(Y)$ is del Pezzo.

Theorem (Choe-K)

- Let C be a curve in $\mathbb{P}^N$. $S^q(C)$ is del Pezzo if and only if C is del Pezzo.
- If $\mathrm{Sec}^q(X)$ is a del Pezzo then $\mathrm{Sec}^{q-1}(X_{\mathbb{T}_x(X)})$ is also a del Pezzo.
- Therefore, If $S^q(X)$ is a del Pezzo q secant variety then the general tangential projection of X at general points $x_1, x_2, \ldots, x_{q-1}$ is also del Pezzo.

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Syzygy structure of secant varieties of minimal degree

Theorem (Choe-K)

- $\mathrm{Sec}^q(X)$ is of minimal degree for some $q \geq 1$ if and only if the minimal free resolution of $R/I_{S^q(X)}$ is of the simplest form

$$R \leftarrow R^{\beta_{1,q}}(-q-1) \leftarrow \cdots \leftarrow R^{\beta_{e-1,q}}(-q-e+1) \leftarrow R^{\beta_{e,q}}(-q-e) \leftarrow 0$$

with $\beta_{p,q} = \binom{p+q-1}{q}\binom{e+q}{p+q}$.

- The ideal $I_{S^q(X)}$ is defined as the $(q+1) \times (q+1)$-minors of 1-generic matrix of size $(q+1) \times (e+q)$, $e \geq 1$ with the simplest Betti table:

	0	1	$\cdots$	$e-1$	e
0	1	–	$\cdots$	–	–
q	–	$*$	$\cdots$	$*$	$*$

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Syzygy structure of del Pezzo secant varieties

Theorem (Choe-K)

- $\mathrm{Sec}^q(X)$ is del Pezzo for some $q \geq 1$ if and only if the minimal free resolution of $R/I_{S^q(X)}$ is q-pure Gorenstein of the form

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with $\beta_{p,q} = \binom{p+q-1}{q}\binom{e+q}{p+q} - \binom{e+q-p-1}{q-1}\binom{e+q-1}{e+q-p}, 1 \leq p \leq e-1.$

- $S^q(X)$ has the following q-pure Gorenstein Betti table:

	0	1	$\cdots$	$e-1$	e
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Theorem (Choe-K)

Suppose that $S^q(X)$ is a q-secant variety of minimal degree with codimension $e \geq 2$. Then, $I_{S^q(X)}$ is generated by $(q+1)$-minors of a 1-generic linear matrix M whose 2-minors in I_X, and either

1. *of size $(q+1) \times (e+q)$ (scroll type); or*
2. *symmetric of size $(q+2) \times (q+2)$ with $e = 3$ (Veronese type).*

Example

The determinantal presentation of $S^q(S(a_1, \ldots, a_n))$ is

$$\begin{pmatrix} x_{1,0} & x_{1,1} & \cdots & x_{1,a_1-q} & & x_{n,0} & x_{n,1} & \cdots & x_{n,a_n-q} \\ \vdots & \vdots & & \vdots & \cdots & \vdots & \vdots & & \vdots \\ x_{1,q} & x_{1,q+1} & \cdots & x_{1,a_1} & & x_{n,q} & x_{n,q+1} & \cdots & x_{n,a_n} \end{pmatrix},$$

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The 1-generic linear matrix M whose 2-minors in I_X appearing in the Theorem can be uniquely constructed as follows:

Corollary

Assume further that X is smooth and embedded by $|V| \subseteq |L|$. Then, there is a unique decomposition $L = L_1 \otimes L_2$ with linear systems $|V_i| \subseteq |L_i|$ such that $V_1 \otimes V_2$ maps to V and either

- $\dim |V_1| = q$ *and* $\dim |V_2| = e + q - 1$*; or*
- $|V_1| = |V_2|$ $(L_1 = L_2)$ *with* $\dim |V_i| = q + 1$*.*

Examples

- $H = qF + (H - qF)$ for all smooth rational normal scrolls;
- $H = L + (2L - \sum_{i=1}^{m} E_i)$ for the blowup $\mathrm{Bl}_m(\mathbb{P}^2)$, $0 \leq m \leq 3$; and
- $\mathcal{O}_{\mathbb{P}^n}(2) = \mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathcal{O}_{\mathbb{P}^n}(1)$ for the second Veronese variety $\nu_2(\mathbb{P}^n)$.

Questions

- Tangential projections of projective varieties are very delicate even if a given point is general.
- Their images are part of tangent cones of secant varieties at a general point of projective varieties.

  Basic questions include the following:
  - When the images are varieties of minimal degree or del Pezzo varieties;
  - Relation to classify smooth varieties with a minimal or del Pezzo q-secant varieties.
- Finding more Matryoshka structures for higher secant varieties, i.e. the generalized gonality conjecture, etc.

♣ **Thank you for your attention!**

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